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COMBINED EFFECTS OF SHEAR DEFORMATION AND PERMANENT INDENTATION ON THE IMPACT RESPONSE OF ELASTIC PLATES†

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Abstract—The dynamic response of a shear-deformable elastic plate to the impact of a mass causing permanent indentation has been considered. Experimental data form the basis of the indentation law and an elastic recovery is assumed when the striking mass leaves the plate. The contact force and the plate center displacement time histories as well as the energy absorbed during impact have been calculated. These results have been compared for elastic impacts with and without shear deformation effects.

INTRODUCTION

The impact problems on beams and plates have been studied quite extensively (Goldsmith, 1960). The elastic impact of a mass on such structures has been considered in the classic works of Timoshenko (1913), Karas (1939), Zener (1941). Chattopadhyay (1977) addressed the effect of shear deformation on the impact response of elastic plates using Mindlin (1951) plate theory and concluded that for high velocities such effects are important. In an experimental study, Barnhart and Goldsmith (1957) showed that permanent craters can be produced even at relatively low impact velocities. Chattopadhyay (1987) used experimental data on permanent indentation in order to obtain the dynamic response of large elastic plates. The contact force and the displacement time histories were shown to be significantly altered with the inclusion of permanent indentation effects. Furthermore, additional energy was shown to be dissipated in the permanent indentation process, when compared to the elastic case.

The present work is aimed at comparing the two important mechanisms, namely that of shear deformation and permanent indentation on the impact response of elastic plates. In previous works such effects were individually treated. Because of the nonlinear load deflection relationship at the contact region a simple superposition is not possible to address the two important effects. The motivation of this work is guided by two important considerations, namely (a) better mathematical modeling and (b) practical application involving permanent indentations. First, the shear deformation mechanism provides a good correlation between two-dimensional plate theory and the three-dimensional elastic solutions. Second, permanent indentation effects provide important information on erosion, wear and fatigue, when considering impacting between machine elements.

In this work, the contact force and the plate displacement time histories have been calculated using an experimentally obtained indentation law for the classical plates as well as the shear-deformable Mindlin plates. The results have been compared with the corresponding elastic impact solutions.

In plate impact studies, it is relevant to determine how much energy of the striking mass is absorbed in the plate. An important contribution to the understanding of impact characteristics has been provided by Zener and Feshbach (1939) in their considerations for energy transfer. The actual contact force $F(t)$ is expressed in terms of a normalized force. It has been established by Zener and Feshbach and extended by Lee (1940) that errors resulting from the ignorance of actual contact force can be mostly eliminated in this way.

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The method uses a normalized contact force which is insensitive to the actual description of the contact force. In this work the energy loss during impact has been calculated by using the peak contact force and contact times for elastic and inelastic impacts using classical and shear deformable plate theories.

RECTANGULAR PLATE VIBRATION

The natural frequencies of vibration of rectangular plate should be determined in order to study the plate impact response. For this purpose, we consider the classical plate theory and the Mindlin plate theory which considers shear deformation effects.

(i) *Classical plate theory*

The equation of motion in the absence of external loads is given by

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \tag{1}$$

where w is the transverse displacement, ρ is the mass density, h is the plate thickness and $D = Eh^3/12(1-\nu^2)$, where E is the Young's modulus and ν is Poisson's ratio for the plate material.

From eqn (1) we obtain the frequency equation of a simply supported plate for the (m, n) mode as,

$$\frac{\rho h}{D}\omega_{mn}^2=\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4, \quad \text{for} \quad m,n=1,2,\ldots,\infty. \tag{2}$$

(ii) *Mindlin plate theory*

The equations of motion developed by Mindlin are given by

$$D\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}D\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}D\frac{\partial^2\psi_y}{\partial x\,\partial y}-\kappa^2Gh\psi_x-\kappa^2Gh\frac{\partial w}{\partial x}=\frac{\rho h^3}{12}\frac{\partial^2\psi_x}{\partial t^2}$$

$$\frac{1+\nu}{2}D\frac{\partial^2\psi_x}{\partial x\,\partial y}+\frac{1-\nu}{2}D\frac{\partial^2\psi_y}{\partial x^2}+D\frac{\partial^2\psi_y}{\partial y^2}-\kappa^2Gh\psi_y-\kappa^2Gh\frac{\partial w}{\partial y}=\frac{\rho h^3}{12}\frac{\partial^2\psi_y}{\partial t^2}$$

$$\kappa^2Gh\frac{\partial\psi_x}{\partial x}+\kappa^2Gh\frac{\partial^2 w}{\partial x^2}+\kappa^2Gh\frac{\partial\psi_y}{\partial y}+\kappa^2Gh\frac{\partial^2 w}{\partial y^2}=\rho h\frac{\partial^2 w}{\partial t^2} \tag{3}$$

where ψ_x and ψ_y are the rotations of the plane sections, G is the shear modulus and $\kappa = \pi^2/12$ is a shear correction factor introduced by Mindlin (1951).

We note that the contribution of the rotatory inertia terms is small and can be neglected (Mindlin, 1951). Using this criterion, we obtain the frequency of a simply supported plate for the (m, n) mode as,

$$\omega_{mn}^2=\frac{D\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}{\rho h\left[1-\frac{D}{\kappa^2Gh}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\right]}, \tag{4}$$

For a concentrated force $F(t)$ applied at the center of the plate $(a/2, b/2)$ the displacement response of the plate is given by eqn (5)

$$w(x,y,t) = \frac{4}{abph} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{\omega_{mn}} \int_0^t F(\tau) \sin \omega_{mn}(t-\tau)\, d\tau \times \sin\frac{m\pi}{2} \sin\frac{n\pi}{2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \quad (5)$$

THE IMPACT PROBLEM

The problem will be formulated both for elastic and inelastic impacts. The elastic impact refers to the case considered by Karas (1939) and Zener (1941). In these formulations, Hertzian contact force law is assumed for the effect of local elastic deformation at the region of contact, i.e.

$$F = kS^{3/2} \quad (6)$$

where S is the relative approach of the mass and the plate, and k is a material constant that depends on the radius of the striking mass and the modulus of elasticity of the mass and the plate.

For the case of permanent indentation, the following assumptions have been made similar to Barnhart and Goldsmith (1957).

(i) The initial kinetic energy is completely transformed into the kinetic energy of rebound and the energy needed to form the permanent crater. This requires that the mass deforms only elastically and that vibrations of the mass and plate can be neglected. The elastic deformation of the mass was verified experimentally by Barnhart and Goldsmith (1957). Furthermore, Rayleigh (1906) showed that very little vibration is induced in an oscillating system under the influence of forces of duration long in comparison with its natural periods.

(ii) The rebound energy of the mass is derived solely from the elastic strain energy stored equally in the mass and plate. This has been substantiated by the work of Tabor (1955) for identical elastic properties of the mass and plate.

(iii) The force law is derived from the modified Hertzian law of contact, i.e. for indentation,

$$F = NS^n, \quad 0 \leqslant S \leqslant S_m, \quad (7)$$

and for the recovery process,

$$F = F_m \left(\frac{S - S_r}{S_m - S_r} \right)^{3/2}, \quad S_m \geqslant S \geqslant S_r, \quad (8)$$

when S_m is the maximum relative approach, S_r, is the permanent crate depth, and N and n are material constants to be obtained experimentally. Note in eqn (8), when $S = S_m$, $F = F_m$, the peak contact force. Note the recovery process is Hertzian or elastic.

DETERMINATION OF CONTACT FORCE

The nonlinear integral equations of the contact force for the impact are obtained by combining the relative approach of the striking mass and the plate with the assumed force-deflection relationship at the contact point given by eqns (7) and (8). The solution is effected by the small-time increment method, where the time increment is sufficiently small so that a linear force–time relation may be used during each increment. At the beginning of contact, the local deformation dominates. Therefore, a recursive scheme that neglects the structural deflection in the initial stage has been formulated for the solution of the contact force. This scheme is shown to be convergent for both elastic and inelastic contacts.

For the recovery process the solution starts off with the initial conditions on force and displacement corresponding to the time when the contact force reaches a maximum value, obtained from the indentation process.

ENERGY TRANSFER DURING IMPACT

The total energy absorbed in the (m, n) mode of the plate is given by

$$\Delta E_{mn} = (2/\rho abh)\left|\int_0^{T_c} F(\tau)\,\mathrm{e}^{i\omega_{mn}\tau}\,\mathrm{d}\tau\right|^2 \tag{9}$$

where T_c is the time when the contact ceases. Following Zener and Feshbach (1939) we normalize the contact force as

$$f(t) = F(t)\Big/\int_0^{T_c} F(\tau)\,\mathrm{d}\tau. \tag{10}$$

This normalization procedure minimizes the errors resulting from the ignorance of the actual contact force. If e denotes the coefficient of restitution, then the change of momentum produced by the contact force is

$$m_0 v_0(1+e) = \int_0^{T_c} F(\tau)\,\mathrm{d}\tau. \tag{11}$$

Denoting E, the initial energy of the mass, by $m_0 v_0^2/2$, we have

$$\Delta E/E = (1+e)^2 R \tag{12}$$

where

$$R = (4m_0/\rho abh)\sum_{m=1,3,\ldots}^{r}\ \sum_{n=1,3,\ldots}^{r}\left|\int_0^{T_c} f(\tau)\,\mathrm{e}^{i\omega_{mn}\tau}\,\mathrm{d}\tau\right|^2. \tag{13}$$

From conservation of energy we have

$$\Delta E/E = 1-e^2. \tag{14}$$

From eqns (12) and (14) we have

$$e = (1-R)/(1+R). \tag{15}$$

NUMERICAL RESULTS AND DISCUSSION

Consider a rectangular steel plate with the dimensions $a = b = 30$ in. (76.2 cm) and $h = 1/2$ in. (1.27 cm) upon which a steel sphere of diameter 1/2 in. (1.27 cm) impinges with a velocity of 150 ft s^{-1} (4572 cm s^{-1}). Barnhart and Goldsmith (1957) report experimental results for the case of the impact of an identical projectile on a beam of 30 in. (76.2 cm) span and a cross-section of 1/2 in. × 1/2 in. (1.27 cm × 1.27 cm). In our plate impact study, we assume an identical local force indentation law. The indentation parameters used in our analysis are $k = 10.989 \times 10^6$, $N = 1.29 \times 10^6$, and $n = 1.128$. The crater depth in the plate is also based on the experimental values for the beam from Barnhart and Goldsmith (1957). In the numerical computations, we employ nondimensional quantities. We define

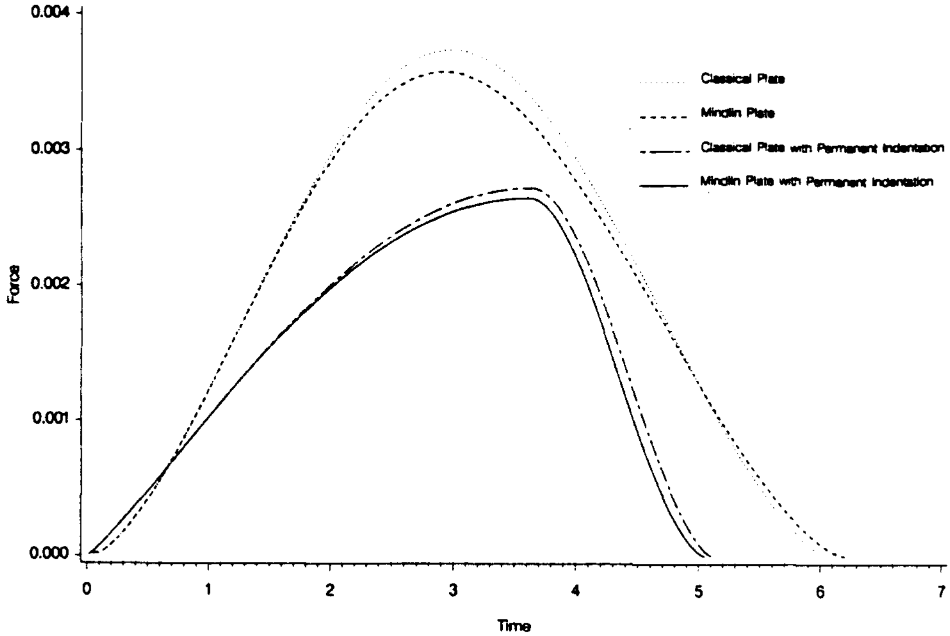

Fig. 1. Contact force variation.

dimensionless time,

$$t^* = t/(\rho h^{5/2}/k)^{1/2}, \tag{16}$$

dimensionless contact force,

$$F^* = F/kh^{3/2}, \quad \text{and} \tag{17}$$

dimensionless central deflection

$$w^* = w(a/2, b/2, t)/h. \tag{18}$$

The striking elastic sphere is taken to have a dimensionless mass

$$m^* = m/\rho h^3 \tag{19}$$

and a dimensionless velocity

$$V^* = V/(E/\rho)^{1/2}. \tag{20}$$

The contact force variation for various cases is shown in Fig. 1. The combined effect of shear deformation and permanent indentation is found to change the nature of the contact force variation and to decrease the maximum value of the contact force. The contact times for the cases involving permanent indentation (with and without shear deformation) are somewhat less when compared with the corresponding elastic solutions.

Figure 2 shows the plate central displacement as a function of time for various cases. All the curves seem to follow the same pattern with displacement steadily increasing, then reaching a plateau and then increasing to a peak value when the waves get reflected off the boundary of the plate. If the plate were infinite, the displacement corresponding to the

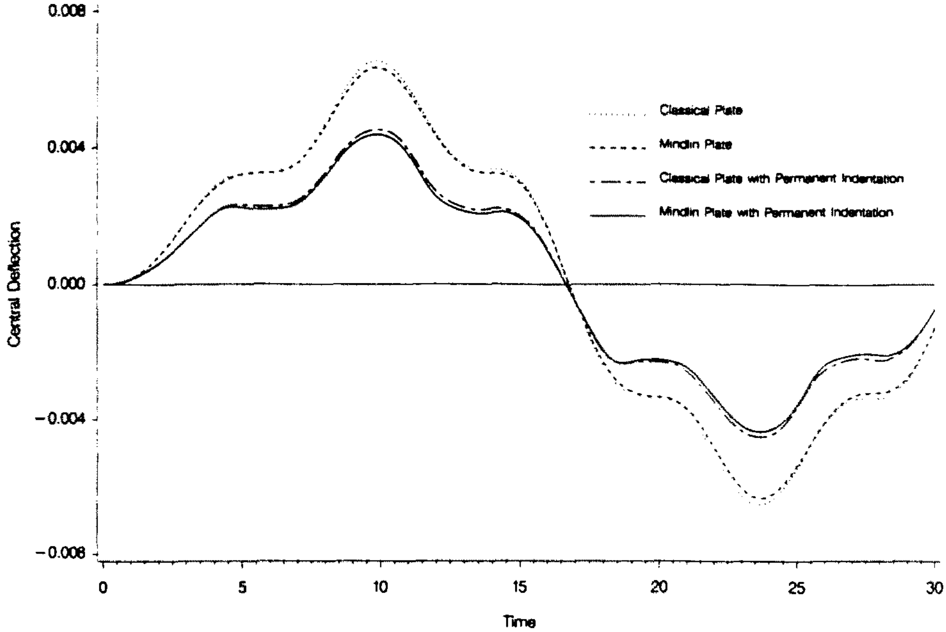

Fig. 2. Plate central deflection.

steady value (at the plateau) would continue indefinitely which has been identified as "intrinsic inelasticity" of large plates by Zener (1941). The peak displacements associated with the permanent indentations are significantly less compared to the elastic impact solutions because of the smaller contact force resulting at the contact region.

Figures 1 and 2 indicate the minimal contribution of shear deformation as contact force and displacement time histories. The effect is not pronounced at moderate impact velocities. For higher impact velocities significant differences can result due to shear deformation as reported by Chattopadhyay (1977).

Figure 3 shows the energy absorbed in the plate as a fraction of the initial kinetic energy of the mass for the various cases considered. Typically, the energy absorbed associated with

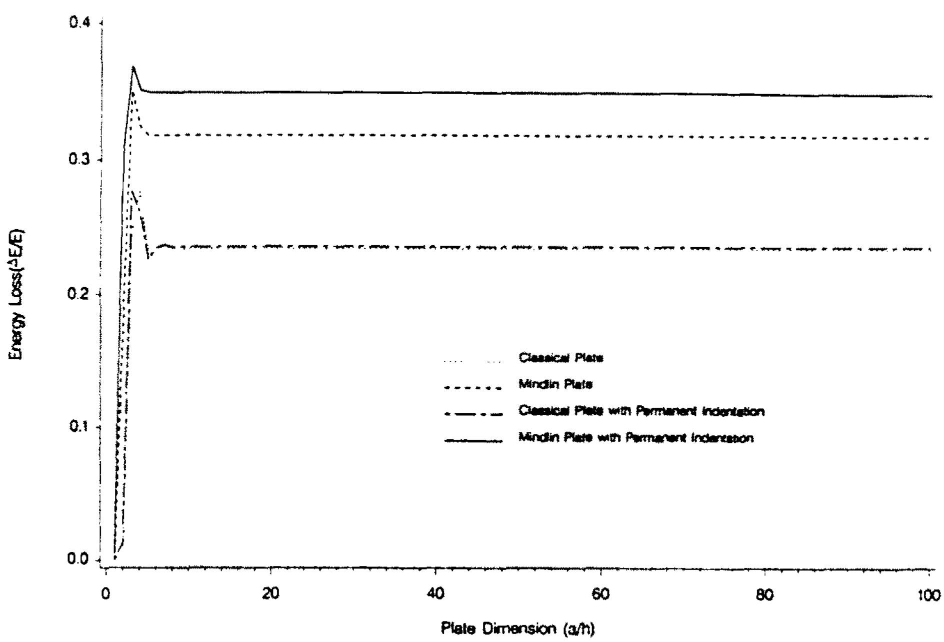

Fig. 3. Energy absorption as a function of plate size.

permanent indentation is more compared to elastic impact, with additional energy going to deform the plate permanently. However, more spectacular is the case involving indentation in a shear-deformable plate. Considerably more energy is absorbed in a shear-deformable plate when considering permanent indentation, than for the case of elastic deformation alone. A major contribution comes from the dynamic plasticity where shear deformation effects are pronounced, as shown by Jones and Gomes de Oliveira (1979).

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